Optimization of the Improved Nyquist Filter With a Piece-Wise Rectangular Characteristic

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***Abstract*—We proposed a new parametric method for the design of the Nyquist filter using constraints in the frequency characteristics construction. We studied the family of new Nyquist filters based on a composite frequency characteristic and investigated its performances in terms of ISI error probability. This method of constructions allows us to study the behavior of the pulses produced by Nyquist filters in terms of ISI error probability. We also observed the variation of the side lobes of the pulses when a larger number of the parameter are introduced in the proposed filter frequency characteristic.**

***Index Terms*—intersymbol-interference (ISI), intercarrier-interference (ICI), frequency offset, Nyquist filter**

I. INTRODUCTION

According to Nyquist first criterion, to ensure *intersymbol interference* elimination it is necessary that the pulse shape should present zero crossing at the ideal sampling moments, excepting the current one and the amplitude should decay rapidly outside the bit interval. The rectangular function appears to be desirable in minimizing the bandwidth used, but is awkward in terms of its slow decay in frequency. Its frequency response is $\sin c(x) = [\sin(p\,x)]/p\,x$, which has infinite tails and the rectangular filter characteristic cannot be implemented.

Nyquist proved that one can use any type of low-pass filter with odd symmetry about the corresponding ideally band-limited cut-off frequency f_o to replace the theoretically ideal characteristic, physically unrealizable (Fig. 1).

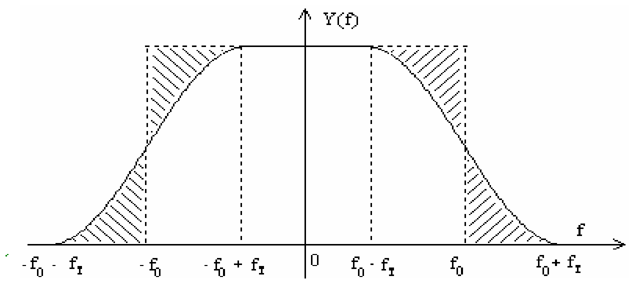


Figure 1. Equivalence between the ideal low-pass filter and the Nyquist filter.

In the sequel we propose to study the behavior of the low-pass filter with odd symmetry about the corresponding ideally band-limited cut-off frequency that is defined for positive frequencies by eq. (1). The characteristic is built combining rectangular functions, as shown in figure 2. The transfer function presents $2k+2$ intervals where is constant over frequency intervals of equal width.

The characteristic is built so that the shaded surfaces obeying odd-symmetry in figure 2 are equal.

$$X(f)=\begin{cases} 1, & |f|\le(1-a)B \\ a_1, & (1-a)B\le|f|\le\left(1-\frac{2k-1}{2k+1}a\right)B \\ \mathbf{L} & \\ a_k, & \left(1-\frac{3}{2k+1}a\right)B\le|f|\le\left(1-\frac{1}{2k+1}a\right)B \\ 1/2, & \left(1-\frac{1}{2k+1}a\right)B\le|f|\le\left(1+\frac{1}{2k+1}a\right)B \\ 1-a_k, & \\ \mathbf{L} & \\ 1-a_1, & \left(1+\frac{2k-1}{2k+1}a\right)B\le|f|\le(1+a)B \\ 0, & |f|>(1+a)B \end{cases} \quad (1)$$

The function $X(f)$ was chosen in order to satisfy:

$$X((1-a)B)=1 \text{ and } X(B)=1/2 \quad (2)$$

Figure 2 illustrates the filter frequency characteristic defined by eq. (1) for positive frequencies.

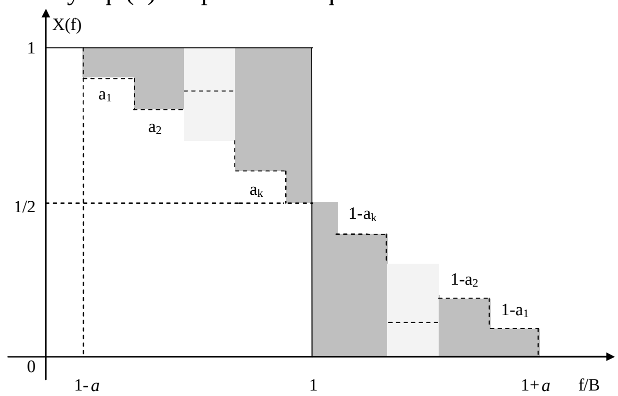


Figure 2. Proposed filter frequency characteristic.

II. PROPOSED CHARACTERISTICS

In the sequel, we study the behaviour of the proposed pulses produced by the low-pass filter defined with relation (2), in terms of the inter-symbol interference for three particular cases.

Case 1.

In the first case we consider k = 1 and the characteristic's shape is defined by the following equation:

$$X_1(f)=\begin{cases}1, & |f|\le(1-a)B\\ a_{1,} & (1-a)B\le|f|\le\left(1-\frac{1}{3}a\right)B\\ 1/2, & \left(1-\frac{1}{3}a\right)B\le|f|\le\left(1+\frac{1}{3}a\right)B\\ 1-a_1 & \left(1+\frac{1}{3}a\right)B\le|f|\le(1+a)B\\ 0, & (1+a)B>|f|\end{cases}\qquad(3)$$

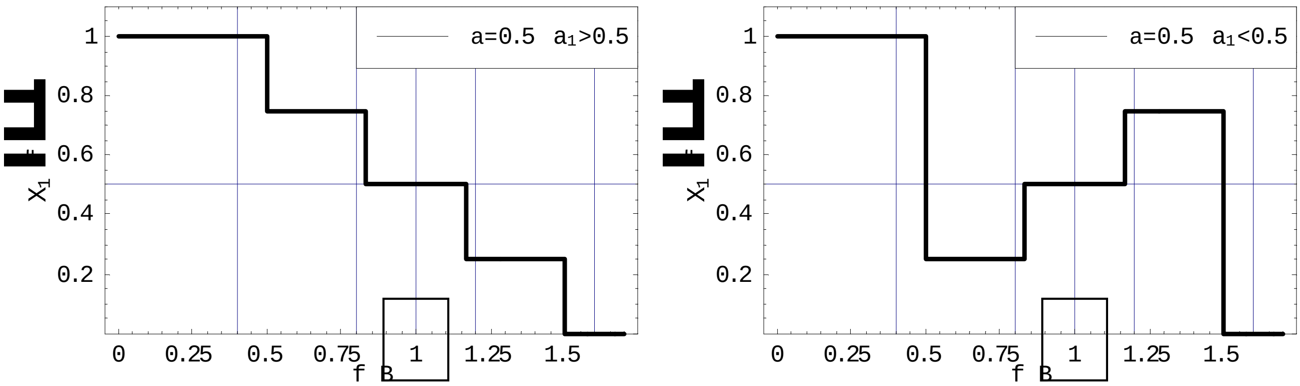
We observe that the function $X_1(f)$ is constant on four intervals and the precise shape of the proposed pulse spectrum is determined by the parameters a and a_1. Figure 3 illustrates the filter characteristics for different values of a_1.

The impulse response is represented in figure 4 for different values of the roll-off factor a and the parameter a_1. We observe that for the case when $a_1=0.75$ (figure 4.a.) an increase of the roll-off factor a determines the time domain tails to decay rapidly. In figure 4.b. ($a_1=0.25$) the increase of the roll-off factor a results in a slowest rate of decay in time domain.

Case 2.

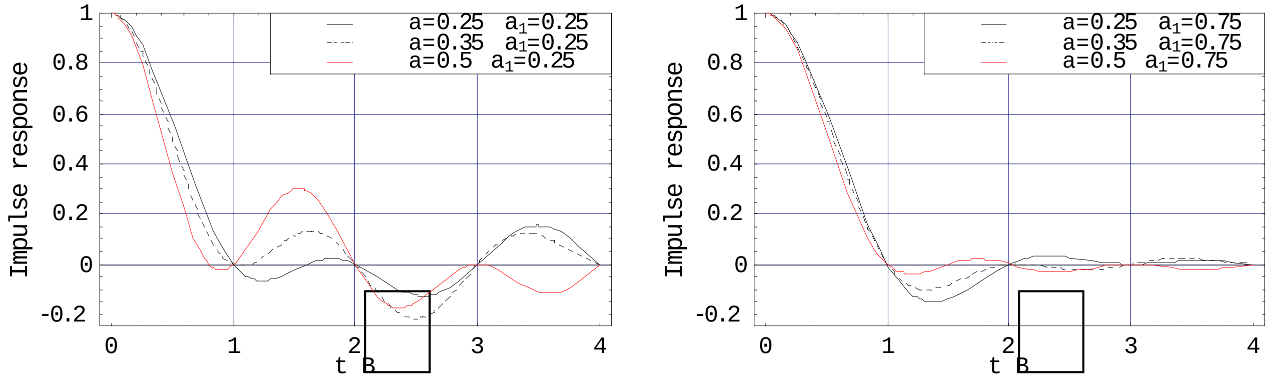
In this case we consider k = 2 and the resulting characteristic is obtained using relation (4). We observe that the function $X_2(f)$ is constant over six intervals and is built using the parameters a_1 and a_2.

$$X_2(f)=\begin{cases}1, & |f|\le(1-a)B\\ a_1, & (1-a)B\le|f|\le\left(1-\frac{3}{5}a\right)B\\ a_2 & \left(1-\frac{3}{5}a\right)B\le|f|\le\left(1-\frac{1}{5}a\right)B\\ 1/2, & \left(1-\frac{1}{5}a\right)B\le|f|\le\left(1+\frac{1}{5}a\right)\\ 1-a_2, & \left(1+\frac{1}{5}a\right)B\le|f|\le\left(1+\frac{3}{5}a\right)B\\ 1-a_{1,} & \left(1+\frac{3}{5}a\right)B\le|f|\le(1+a)B\\ 0, & |f|>(1+a)B\end{cases}$$



a) b)

Figure 3. Frequency characteristics $X_1(f)$: a) $a_1=0.75$, b) $a_1=0.25$.



a) b)

Figure 4. Impulse responses (case 1 - $X_1(f)$) for different values of the roll-off a.
a) $a_1=0.25$, b) $a_1=0.75$.

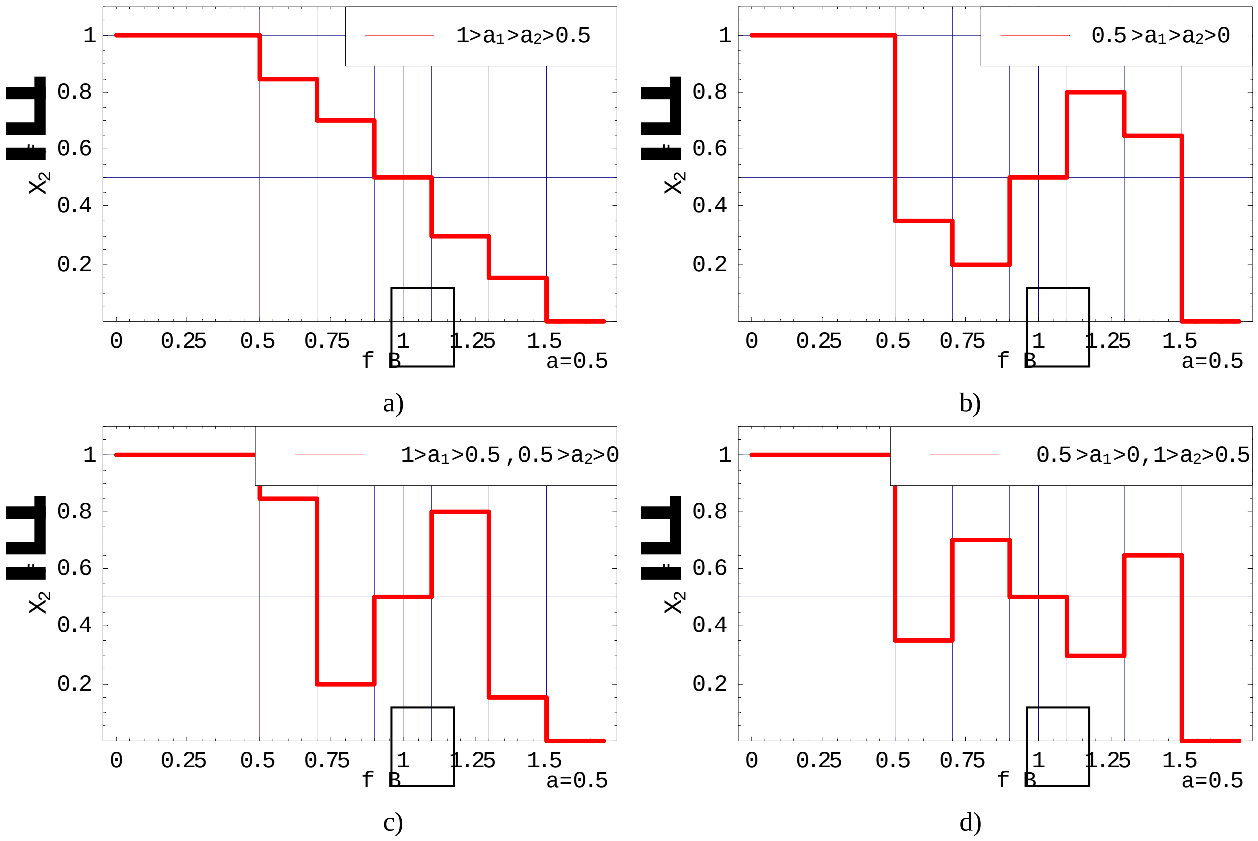


Figure 5. Frequency characteristics $X_2(f)$.

a) $a_1 = 0.85$, $a_2 = 0.75$ b) $a_1 = 0.25$, $a_2 = 0.75$, c) $a_1 = 0.75$, $a_2 = 0.75$, d) $a_1 = 0.75$, $a_2 = 0.75$.

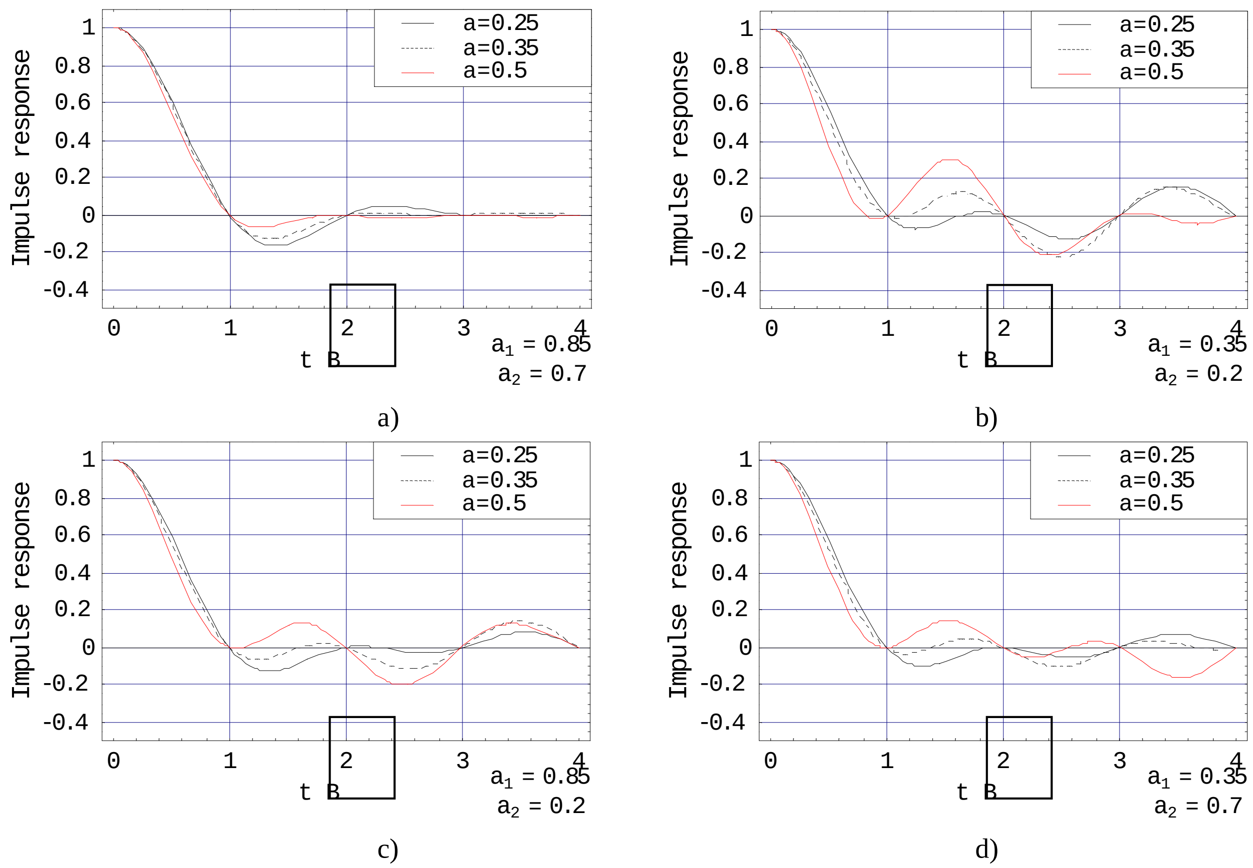


Figure 6. Impulse responses (case 2 - $X_2(f)$) for different values of the roll-off a and parameters a_1 and a_2.

In figure 5 it is presented the filter characteristic when we modify the parameters a_1 and a_2 which interfere in the design of the transfer function for the roll-off factor $a = 0.5$.

The impulse response is represented in figure 6 for several values of the roll-off factor a and the parameters a_1 and a_2. As in the previous case we observe that an increase of the roll-off factor a determines a decrease of the first side lobes in time domain, only if the values of the parameters obey the rule $1 > a_1 > a_2 > 0.5$. For other examples plotted in figure 6, the pulses have an opposite behavior.

Case 3

In the third case we consider $k = 3$ and the resulting characteristic is obtained using relation (4). In this case we have three parameters a_1, a_2 and a_3 used for the construction of filter characteristic. The transfer function $X_3(f)$ is constant over eight intervals.

A look at the figure 8, which presents the impulse response for the third case proposed, reveals the fact that for $1 > a_1 > a_2 > a_3 > 0.5$ the first side lobes decrease rapidly when the roll-off factor a increases.

$$X_3(f) = \begin{cases} 1, & |f| \le (1-a)B \\ a_1, & (1-a)B \le |f| \le \left(1-\frac{5}{7}a\right)B \\ a_2 & \left(1-\frac{5}{7}a\right)B \le |f| \le \left(1-\frac{3}{7}a\right)B \\ a_3, & \left(1-\frac{3}{7}a\right)B \le |f| \le \left(1-\frac{1}{7}a\right)B \\ 1/2, & \left(1-\frac{1}{7}a\right)B \le |f| \le \left(1+\frac{1}{7}a\right)B \\ 1-a_3, & \left(1+\frac{1}{7}a\right)B \le |f| \le \left(1+\frac{3}{7}a\right)B \\ 1-a_2 & \left(1+\frac{3}{7}a\right)B \le |f| \le \left(1+\frac{5}{7}a\right)B \\ 1-a_{1,} & \left(1+\frac{5}{7}a\right)B \le |f| \le (1+a)B \\ 0, & |f| > (1+a)B \end{cases} \qquad (5)$$

From the examples given previously we find out that the increase of the excess bandwidth induces the time domain tails to decay rapidly only if the characteristics have a *stair shape* with steps that are evenly distributed.

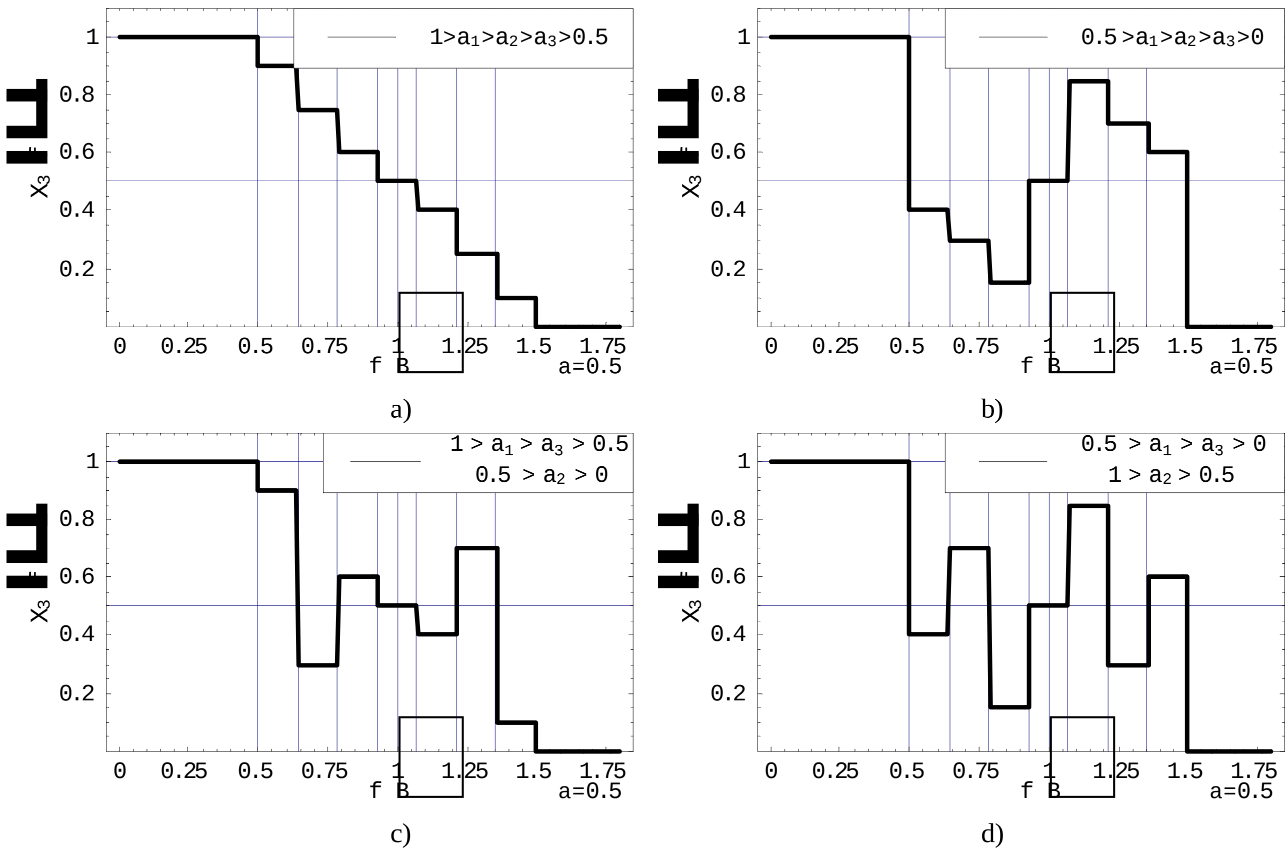


Figure 7. Frequency characteristics $X_3(f)$.

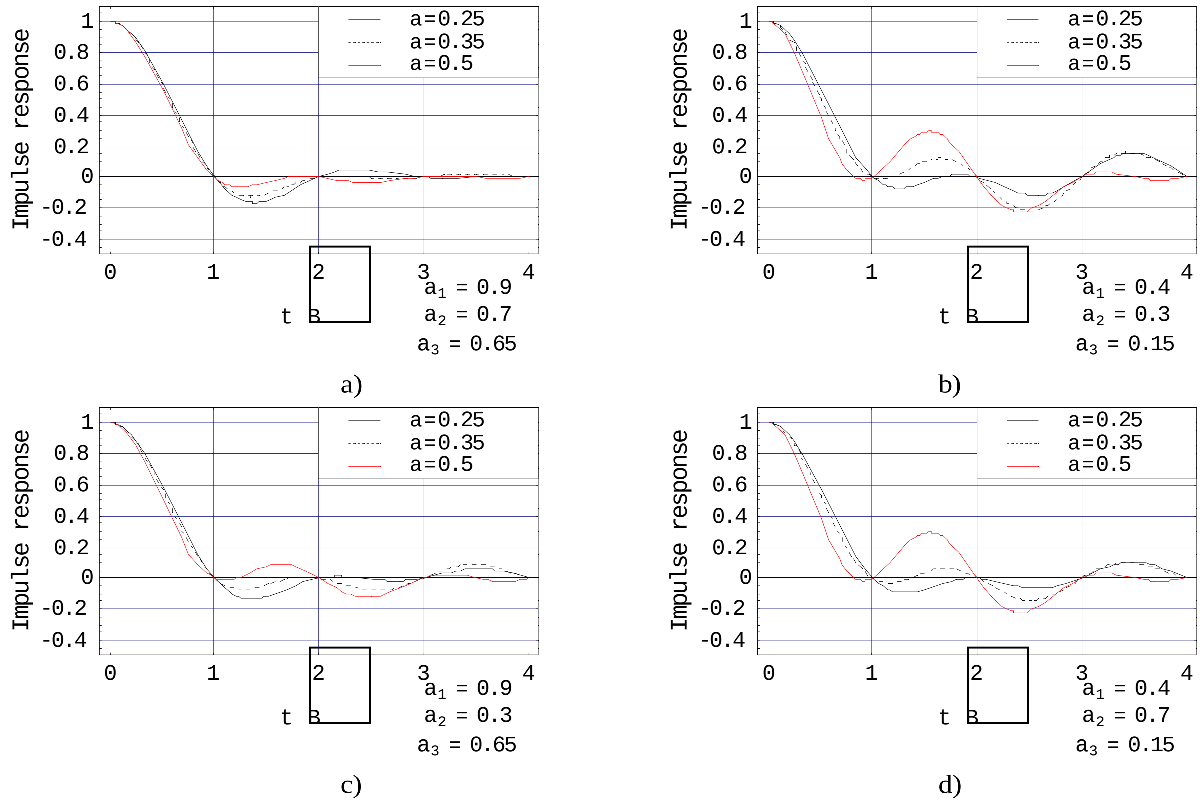


Figure 8. Impulse responses (case 3 - $X_3(f)$) for different values of the roll-off factor a and parameters a_1, a_2.

III. SIMULATION RESULTS

In this section, the pulses produced by the low-pass filters previously mentioned specified are studied in terms of ISI error probability.

The error probability measures the performances of the pulses regarding inter-symbol interferences and includes the effects of noise, synchronization error and distortion. The error probability P_e was evaluated as in [15] using Fourier series.

$$P_e = \frac{1}{2} - \frac{2}{p}\sum_{\substack{m=1\\ M\,odd}}^{M}\left(\frac{\exp\left(-m^2w^2/2\right)\sin\left(mwg_0\right)}{m}\right)\prod_{k=N_1}^{N_2}\cos\left(mwg_k\right) \qquad (6)$$

Here M represents the number of coefficients considered in the approximate Fourier series of noise complementary distribution function; $w = 2p/T_f$ -angular frequency; T_f is the period used in the series; *N1* and *N2* represent the number of interfering symbols before and after the transmitted symbol; $g_k = p(t-kT)$ where *p(t)* is the pulse shape used and *T* is the bit interval. The results are computed using $T_f = 40$ and $M = 61$ for $N=2^{10}$ interfering symbols and SNR = 15 dB, for all the cases.

It is desirable to determine the values of the parameters $a_1,\ldots,a_k$ so that the error probability should be minimized.

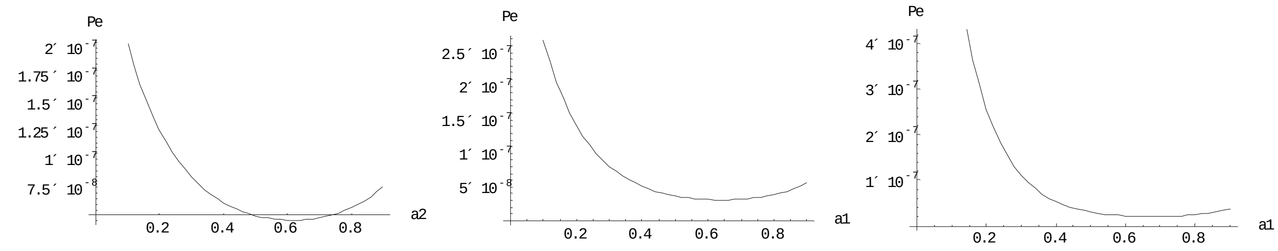
The ISI error probability is calculated using the method in [15] in the presence of time sampling errors for pulses produced by $X_1(f)$ for different values of parameter a_1 and roll-off factor a. In figure 9 is illustrated the variation of error probability when a_1 takes values between 0.1 and 0.9, for timing offset (t/T_B) equal with 0.05 and excess bandwidth a = 0.25, 0.35, and 0.5.

We observe that there exists a minimum value for error probability. In Table 1 are listed the minimum values found for error probability. In order to obtain these results we set roll-off factor a = 0.25, 0.35, and 0.5 and timing offset t/T_B = 0.05, 0.1, and 0.2.

As expected, the error probability is reduced if the roll-off factor increases.

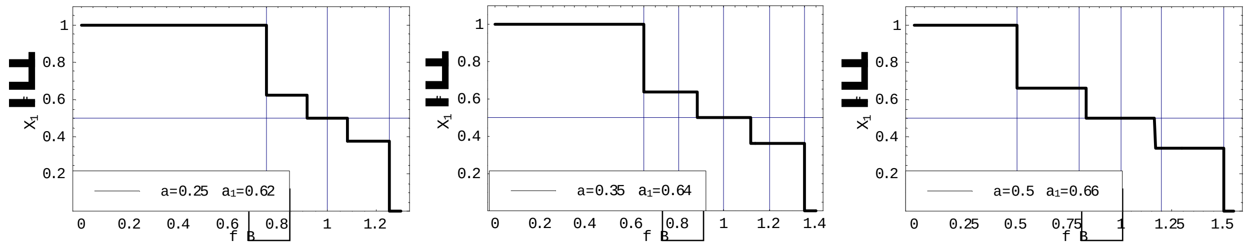
TABLE 1 ISI ERROR PROBABILITY OF THE PROPOSED NYQUIST PULSES FOR N=2^{10} INTERFERING SYMBOLS AND SNR = 15 DB

X_1	$t/T_B = 0.05$		$t/T_B = 0.1$		$t/T_B = 0.2$	
	a_1	P_e	a_1	P_e	a_1	P_e
$a = 0.25$	0.62	4.55572*10^{-8}	0.64	8.42322*10^{-7}	0.66	2.13259*10^{-4}
$a = 0.35$	0.64	3.09292*10^{-8}	0.68	3.78826*10^{-7}	0.7	7.36233*10^{-5}
$a = 0.5$	0.68	2.0085*10^{-8}	0.68	1.44*10^{-7}	0.72	2.32392*10^{-5}



a) $a = 0.25$; b) $a = 0.35$ c) $a = 0.5$

Figure 9. Error probability; $X_1(f)$, $t/T_B = 0.05$.



a) **b)** **c)**

Figure 10. Frequency characteristics $X_1(f)$.

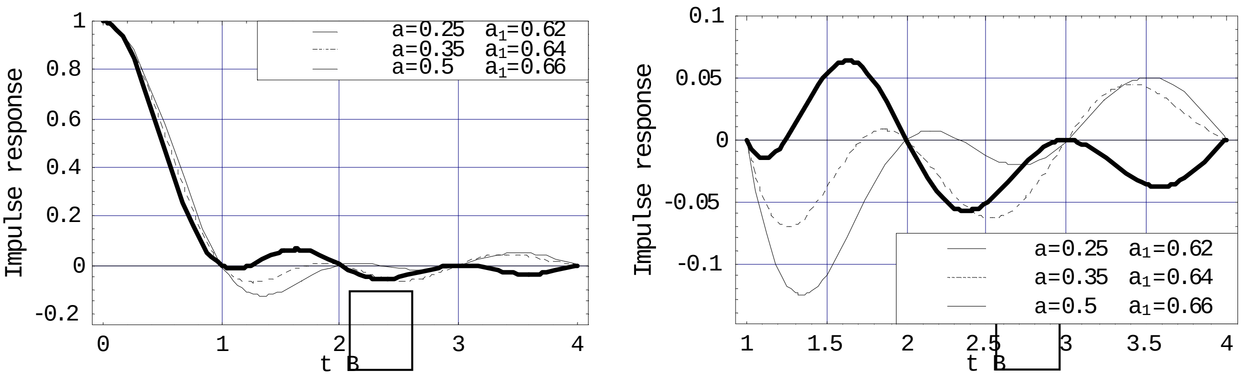
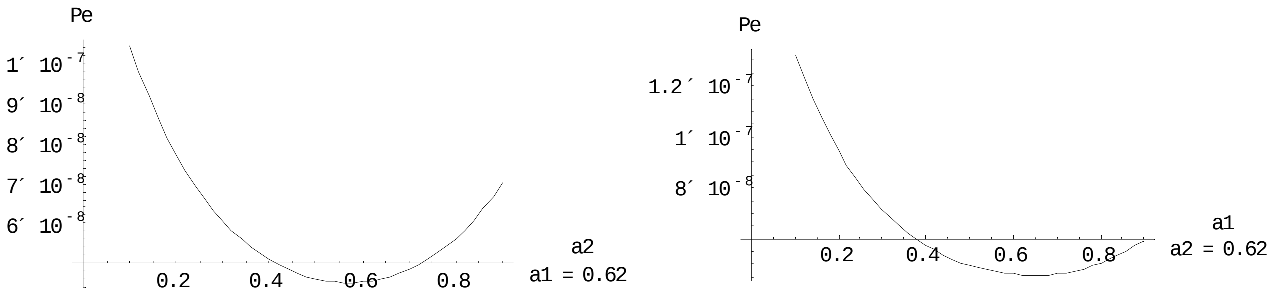


Figure 11. Impulse responses (case 1 - $X_3(f)$.

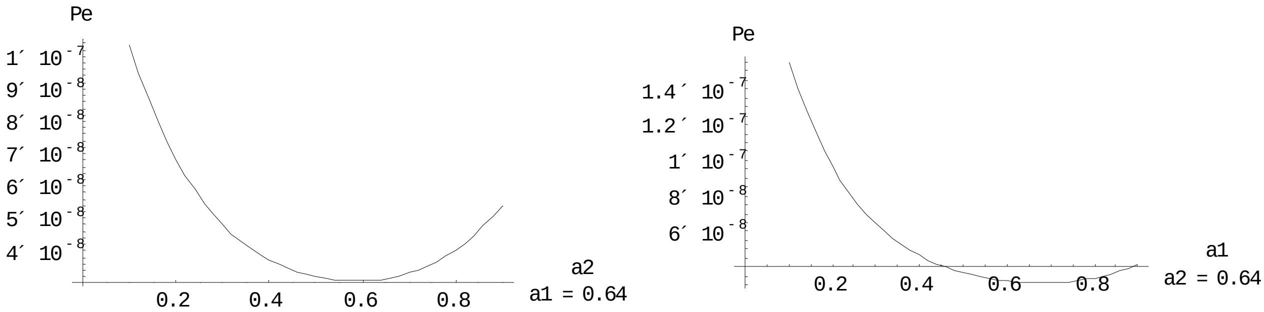
In figure 10 and 11 are plotted the frequency characteristics and impulse responses, respectively, for the proposed shapes and different values of the excess bandwidth, $a = 0.25, 0.35, \text{and } 0.5$. For the parameter a_1 we assign the values found for the minimum error probability and $t/T_B = 0.05$. From figure 11 we observe that the impulse response presents a rapid decay when $a = 0.5$ and the behavior is more flat around $t/B = 3, 4, \ldots$. This fact is reflected in the results presented for the error probability when sampled with a small time offset.

Further we study the case when *k* = 2 and there are two parameters a_1 and a_2. In the previous case we found the values for a_1, when the error probability is minimum. In this case we maintain the value found for the stages and we add a new level. The results for error probability are presented in figure 12. In the interval $\left[(1-a)B, B\right]$ we have three levels of the *stair shape* characteristic: a_1, a_2 and 0.5. For the graphics from the first column (figure 12) we set for parameter a_1 the values obtained in previous case and varied the parameter a_2 between 0.1 and 0.9. In the second column we set the values found the case 1 for the parameter a_2 and varied a_1 between 0.1 and 0.9. The error probability results are plotted in figure 12 only for the case $t/T_B = 0.05$ and reported in Table 2 for the sampling moments $t/T_B = 0.1$ and $t/T_B = 0.2$. We observe an improvement of error probability especially for the cases when $t/T_B = 0.2$.

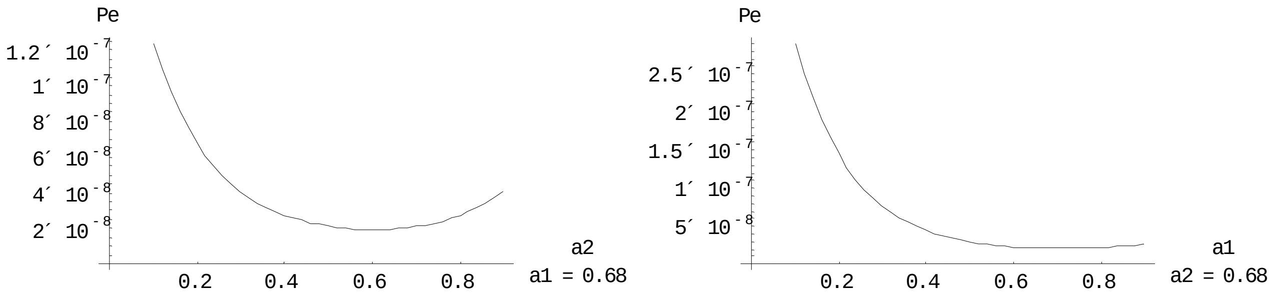
In figure 13 and 14 are plotted the frequency characteristics and impulse responses, for the proposed shapes and different values of the excess bandwidth, $a = 0.25, 0.35, \text{ and } 0.5$ and values of parameters a_1 and a_2 listed in Table 2 with bold characters.



a) $a = 0.25$;



b) $a = 0.35$

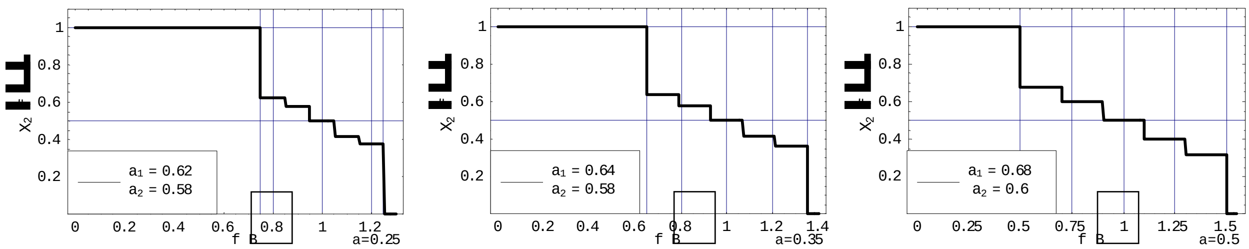


c) $a = 0.5$

Figure 12. Error probability; $X_2(f)$, $t/T_B = 0.05$.

TABLE 2. ISI ERROR PROBABILITY OF THE PROPOSED NYQUIST PULSES FOR $N = 2^{10}$ INTERFERING SYMBOLS AND SNR = 15 DB

X_2	$t/T_B = 0.05$			$t/T_B = 0.1$			$t/T_B = 0.2$		
	a_1	a_2	P_e	a_1	a_2	P_e	a_1	a_2	P_e
$a = 0.25$	**0.62**	**0.58**	**4.53042*10^-8**	0.64	0.58	$8.26096*10^{-7}$	0.66	0.6	$2.03858*10^{-4}$
	0.64	0.62	$4.56569*10^{-8}$	0.68	0.64	$8.37351*10^{-7}$	0.7	0.66	$2.13475*10^{-4}$
$a = 0.35$	**0.64**	**0.58**	**$3.05974*10^{-8}$**	0.68	0.6	$3.56887*10^{-7}$	0.7	0.62	$6.4535*10^{-5}$
	0.68	0.64	$3.0947*10^{-8}$	0.72	0.68	$3.85874*10^{-7}$	0.74	0.7	$7.5798*10^{-5}$
$a = 0.5$	**0.68**	**0.6**	**$1.9494*10^{-8}$**	0.68	0.6	$1.33245*10^{-7}$	0.72	0.62	$1.8073*10^{-5}$
	0.72	0.68	$2.02923*10^{-8}$	0.72	0.68	$1.3952*10^{-7}$	0.78	0.72	$2.2561*10^{-5}$



a) b) c)

Figure 13. Frequency characteristics $X_2(f)$.

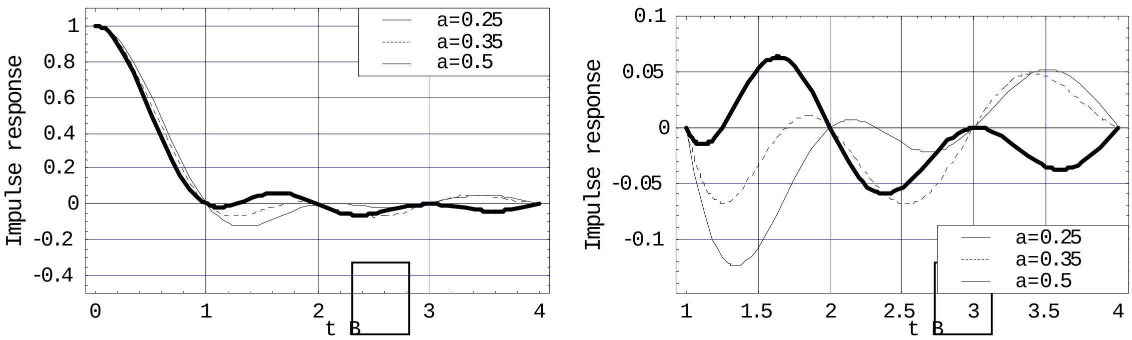
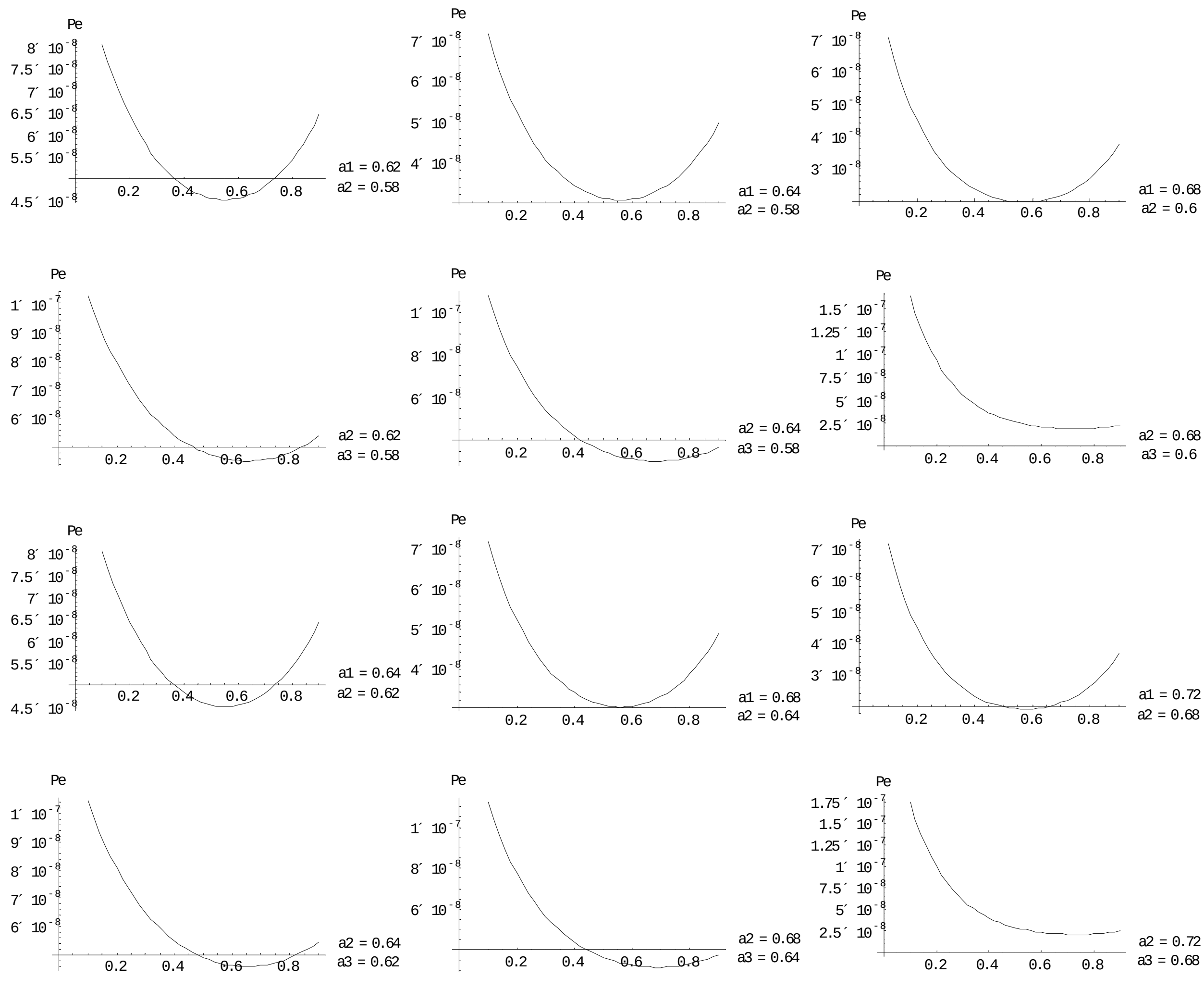


Figure 14. Impulse responses (case 2 - $X_2(f)$).

For the third case we have preserved the values of the stage levels found previously for minimum error probability reported in Table 2 and have inserted a new one. Now the frequency characteristic is built using three parameters a_1, a_2 and a_3. We varied the values of parameters a_1, a_2 and a_3 for different values of excess bandwidth in the presence of sampling errors. The results are illustrated in figure 15 and Table 3.

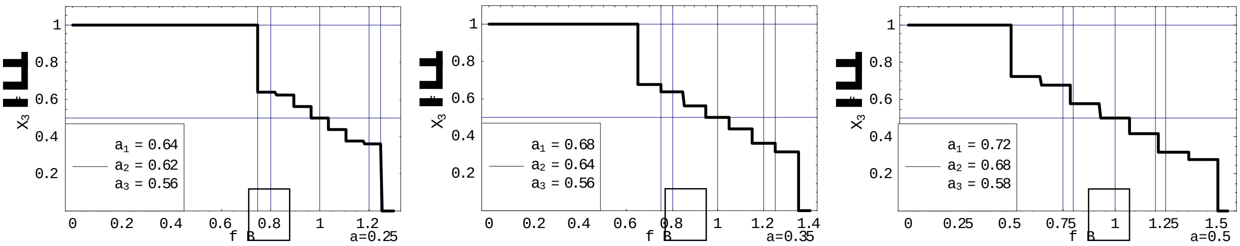


a) $a = 0.25$ b) $a = 0.35$ c) $a = 0.5$

Figure 15. Error probability; $X_3(f)$, $t/T_B = 0.05$.

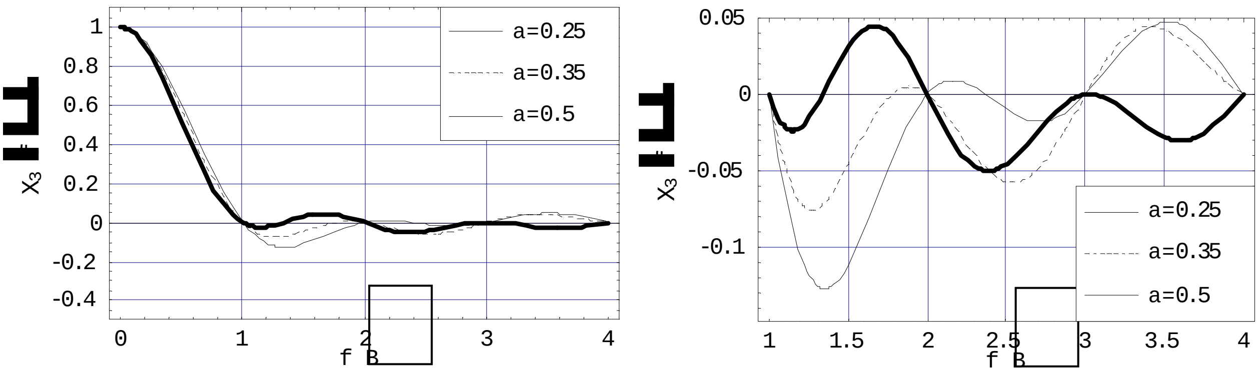
TABLE 3 ISI ERROR PROBABILITY OF THE PROPOSED NYQUIST PULSES FOR $N = 2^{10}$ INTERFERING SYMBOLS AND SNR = 15 DB

X_3	$t/T_B = 0.05$				$t/T_B = 0.1$				$t/T_B = 0.2$			
a	a_1	a_2	a_3	P_e	a_1	a_2	a_3	P_e	a_1	a_2	a_3	P_e
0.25	0.62	0.58	0.56	4.53066*10^-8^	0.64	0.58	0.56	8.35364*10^-7^	0.66	0.6	0.56	2.0598*10^-4^
	0.66	0.62	0.58	4.52765*10^-8^	0.7	0.64	0.58	8.08218*10^-7^	0.72	0.66	0.6	1.98503*10^-4^
	0.64	**0.62**	**0.56**	**4.51854*10^-8^**	0.68	0.64	0.56	8.07038*10^-7^	0.7	0.66	0.58	1.96791*10^-4^
	0.66	0.64	0.62	4.59538*10^-8^	0.7	0.68	0.64	8.63206*10^-7^	0.72	0.7	0.66	2.26877*10^-4^
0.35	0.64	0.58	0.56	3.0741*10^-8^	0.68	0.6	0.58	3.63144*10^-7^	0.7	0.62	0.58	6.59249*10^-5^
	0.68	0.64	0.58	3.03612*10^-8^	0.74	0.68	0.6	3.5026*10^-7^	0.76	0.7	0.62	6.14411*10^-5^
	0.68	**0.64**	**0.56**	**3.03211*10^-8^**	0.72	0.68	0.58	3.47997*10^-7^	0.74	0.7	0.58	6.05711*10^-5^
	0.7	0.68	0.64	3.14924*10^-8^	0.74	0.72	0.68	4.14402*10^-7^	0.76	0.74	0.7	8.70181*10^-5^
0.5	0.68	0.6	0.58	1.96813*10^-8^	0.68	0.6	0.58	1.38056*10^-7^	0.72	0.62	0.58	1.9769*10^-5^
	0.72	0.68	0.6	1.93025*10^-8^	0.76	0.68	0.6	1.21837*10^-7^	0.8	0.72	0.62	1.47727*10^-5^
	0.72	**0.68**	**0.58**	**1.92339*10^-8^**	0.72	0.68	0.58	1.232*10^-7^	0.78	0.72	0.6	1.47408*10^-5^
	0.74	0.72	0.68	2.10606*10^-8^	0.76	0.72	0.68	1.49908*10^-7^	0.82	0.78	0.72	2.98228*10^-5^



a) b) c)

Figure 16. Frequency characteristics $X_3(f)$.



a) b)

Figure 17. Impulse responses (case 3 - $X_3(f)$).

IV. CONCLUSIONS:

The error probability measures the performances of the pulses regarding intersymbol interferences and includes the effects of noise, synchronization error and distortion.

We proposed and evaluated a new type of a *stair shaped* frequency characteristic,, which has $2k+2$ levels in terms of intersymbol interference. From the examples listed above we observe that better results for error probability are obtained when the levels of the stair characteristic are distributed evenly and are placed near the level 0.5.

A look at Table 1, Table 2 and Table 3 reveals the fact that we obtain much better results for error probability when we introduce a new level in the stair shape characteristic, for sampling moment $t/T_B = 0.2$.

As a further work we are going to study a method of interpolation for the frequency characteristic. The method for constructing the filter characteristics will use the piece-wise polynomial approximation.

The pulses obtained and investigated are suitable for OFDM use, to reduce the sensitivity of OFDM systems due to frequency offset.

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