# ISI-FREE PULSES WITH REDUCED SENSITIVITY TO TIMING ERRORS PRODUCED BY LINEAR COMBINATIONS OF FILTERS CHARACTERISTICS

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## ABSTRACT

*This paper presents and investigates a novel approach for constructing family of ISI-free pulses. We propose and discuss a family of new Nyquist pulses obtained from a linear combination of two ISI-free pulses produced by Nyquist filters characteristics. They show reduced sensitivity to timing errors, as compared with some recent pulses introduced in [6].*
***Keywords:*** *intersymbol interference, Nyquist filter, error probability, raised-cosine, flipped-exponential.*

## 1. INTRODUCTION

A classical problem in data communications over bandlimited channels is the search for a pulse shape that produces no intersymbol interference (ISI) and low timing errors.

The most popular variant of Nyquist pulse [1] with wide practical applications is the *raised-cosine* (RC) pulse, which is produced by a low-pass filter with odd symmetry about the corresponding ideally band-limited cut-off frequency and also satisfies Nyquist first criterion. The frequency spectrum is defined as

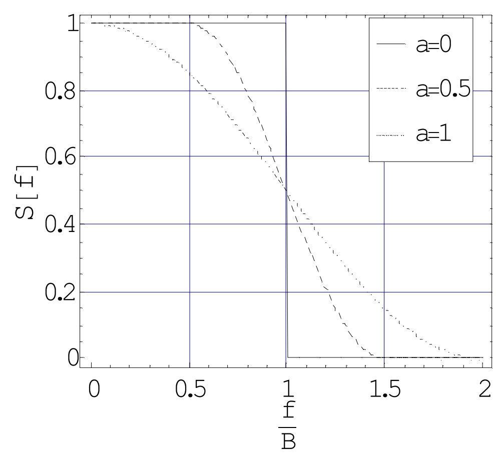
$$S_{RC}=\begin{cases}1, & |f|\le B(1-\alpha)\\ \frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{2B\alpha}\left(|f|-B(1-\alpha)\right)\right), & B(1-\alpha)\le|f|\le B(1+\alpha)\\ 0, & B(1+\alpha)\le|f|\end{cases} \quad (1)$$

The corresponding (scaled) time function is:

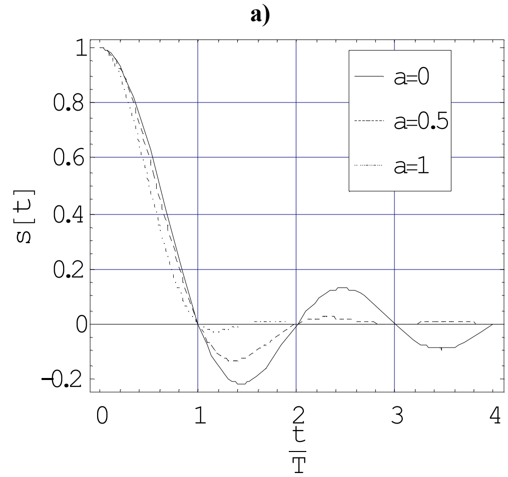
$$p_{RC}=\sin c(t/T)\frac{\cos(2\pi\alpha t/T)}{1-4\alpha^2t^2/T^2} \quad (2)$$

where $B$ is the bandwidth corresponding to symbol repetition rate $T=1/2B$ and $\alpha$ is the roll-off factor and takes values between zero and one. The parameter $\alpha$ also represents the fractional excess bandwith occupied by the signal beyond the Nyquist frequency *1/2T*.

The precise shape of the *raised-cosine* (RC) spectrum is determined by the parameter $\alpha$. A value of $\alpha=0$ reduces *raised-cosine* (RC) pulse to the Nyquist pulse $\sin c(t/T)$ (theoretically ideal pulse) and offers the narrowest bandwidth, but the slowest rate of decay in time domain. When $\alpha=1$ the bandwith is *1/T* but the time domain tails decay rapidly. Thus the parameter $\alpha$ gives a trade-off between increased data rate and time domain tail suppression. (figure 1)

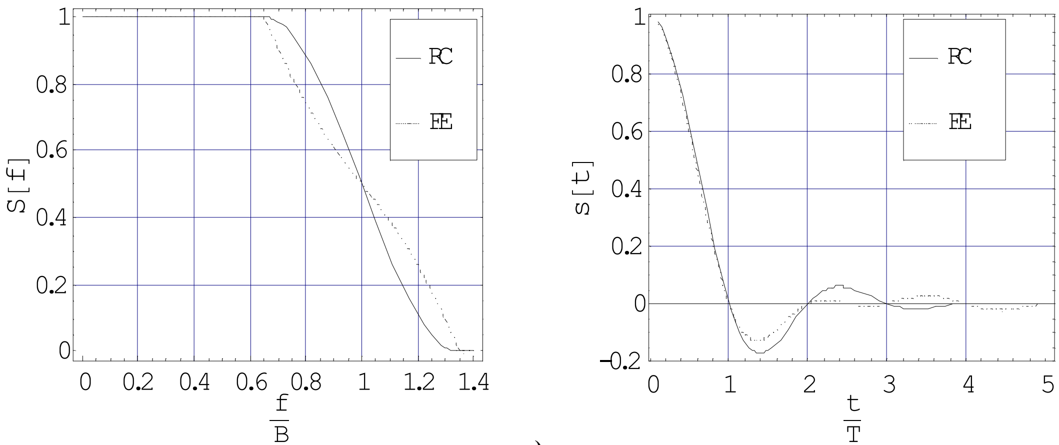



**a)**




**b)**

**Figure 1. Spectral shape (a) and time characteristics (b) of the *raised-cosine* (RC) pulse**




a) b)

**Figure 2. Spectral shape (a) and time characteristics (b) of the *raised-cosine* (RC) pulse and *flipped-exponential* (FE) pulse for an excess bandwidth α =0.35**

Recently, new improved Nyquist pulses that show smaller maximum distortion, more open receiver eye and a smaller symbol error rate in the presence of symbol timing error were reported [2], [3] and [4]. They are defined by equation (3).

$$S_i(f)=\begin{cases}1, & |f|\le B(1-\alpha)\\ G\big((|f|-B(1-\alpha))\big), & B(1-\alpha)\le |f|\le B\\ 1-G\big(B(1+\alpha)-|f|\big) & B<|f|\le B(1+\alpha)\\ 0, & B(1+\alpha)<|f|\end{cases} \quad (3)$$

Where $G(f)$ is a function satisfying $G(0)=1$. In [2],[3] and [4] $G(f)$ was chosen to have a concave shape in the frequency interval $B(1-\alpha)\le|f|\le B$ in order to transfer some energy to the high frequency spectral range. This results in a pulse that decays asymptotically as $t^{-2}$ as compared with $t^{-3}$ for the RC pulse, but with the advantage that the eye diagram is more open and, as a consequence, a better bit error rate is obtained.

Two recent contributions showed that improved Nyquist pulses can be obtained with the *flipped-*$G(f)$ technique, e.g. *flipped-exponential* [2] and *flipped-hyperbolic secant* or *flipped-inverse hyperbolic secant* [3].

The envelope of the impulse response decays as $t^{-2}$ or $t^{-3}$ at best, since the functions and their flipped counterparts are continuous at $f_n=1$.

The first derivative of the *flipped-hyperbolic secant* is continuous at $f_n=1$, which accounts for its steeper decay. The *flipped-exponential* technique uses $G(f)=e^f$ and $\beta=\ln 2/(\alpha B)$, while in [3] $G(f)=\sec h(f)$ and $\beta=\gamma=\ln(\sqrt{3}+2)/(\alpha B)$ or $X(f)=1-\frac{1}{2\alpha B\gamma}\,arc\sec h(f)$ with $\beta=\frac{1}{2\alpha B}$. The first proposed pulse is referred to as *flipped-exponential* (FE). This pulse is also known as *better than raised-cosine* (BTRC) [2], [3]. The frequency spectrum is defined as

$$S_{FE}=\begin{cases}1, & |f|\le B(1-\alpha)\\ \exp\big(\beta(B(1-\alpha)-|f|)\big), & B(1-\alpha)\le|f|<B\\ 1-\exp\big(\beta(|f|-B(1+\alpha))\big), & B<|f|\le B(1+\alpha)\\ 0, & B(1+\alpha)<|f|\end{cases} \quad (4)$$

where $\beta=(\ln 2)/(\alpha B)$.

Its impulse response is given by

$$p_{FE}=\frac{1}{T}\operatorname{sinc}\left(\frac{t}{T}\right)\frac{4\beta\pi t\sin(\pi\alpha t/T)+2\beta^2\cos(\pi\alpha t/T)-\beta^2}{(2\pi t)^2+\beta^2} \quad (5)$$

Figure 2. shows the time representations of *raised-cosine* (RC) pulse compared with the *flipped-exponential* (FE) pulse [2] for α =0.35. Using (2) it can be proved that the tails of the *raised-cosine* (RC) pulse for α >0 decay asymptotically as $t^{-3}$ and using (5) can be proved that the tails of the *flipped-exponential* (FE) pulse for α >0 decay asymptotically as $t^{-2}$. Despite of this fact *flipped-exponential* (FE) pulse is “better than” Nyquist pulse *raised-cosine* (RC). By examinations of figure 2, we observe that the magnitudes of the two largest sidelobes of the *raised-cosine* (RC) pulse are larger than the magnitudes of the two largest sidelobes of the *flipped-exponential* (FE) pulse. In [2] it is reported that the bit error rate in presence of time sampling errors computed using method of [4] are smaller for all values of α and timing offset for the *flipped-exponential* (FE) pulse than the *raised-cosine* (RC) pulse.

The Nyquist pulses proposed in [3]: *flipped-hyperbolic secant* (*fsec*) is better than the *raised-cosine* (RC) pulse and *flipped-inverse hyperbolic secant* (*farcsech*) is better than the *flipped-exponential* (FE) pulse. This new improved parametric pulses are also robust to the root and truncation operations.

A new technique of constructing ISI-free pulses with better performance regarding to error probability has been reported in [6]. There a linear combination of two pulses $p_1(t)$ and $p_2(t)$ was proposed in order to obtain a new pulse $p(t)$ as:

$$p(t) = qp_1(t) + (1-q)p_2(t) \tag{6}$$

The linear combination technique can be applied to any pair of pulses from among *RC*, *FE* and pulses in [3] to obtain new and useful pulses.

The linear combination of two pulses guarantees that the resulting pulse has a bandwidth not greater than that of constituent pulse with larger bandwidth, and if the constituent pulses are ISI-free, then the resulting pulse will also be ISI-free.

In [6] the ISI free pulses: raised-cosine (RC) and flipped-exponential (FE), were linearly combined and optimized using the distribution of timing error. The resulting pulse performs better than raised-cosine (RC) and flipped-exponential (FE) for fixed as well as randomly distributed timing errors while having the same bandwidth

Figure 3. presents the waveforms of the linear combination pulses, raised-cosine (RC) pulse and flipped-exponential (FE) pulse. The better performance of the linear combination pulses is due to the fact that the first two sidelobes are smaller than those of the raised-cosine (RC) pulse and flipped-exponential (FE) pulse. The combination pulse is SNR (signal-to-noise ratio) dependent. In general, the sensitivity of combination pulse to SNR mismatch is a function of α.

In [5] a new class of Nyquist pulses has been proposed, which is defined as

$$G_i(f) = \frac{1}{2B^i\alpha^i}(B-f)^i + \frac{1}{2} \tag{7}$$

For *i* odd they show odd symmetry around *B* and their definition can be

$$S_i = \begin{cases} 1, & |f| \le B(1-\alpha) \\ G(f), & B(1-\alpha) \le |f| \le B(1+\alpha) \\ 0, & B(1+\alpha) < |f| \end{cases} \tag{8}$$

For *i* even, the *flipped- technique* is used and they are denoted *flipped-* $G(f)$ [4].

The impulse responses $s_i(t)$ are given by equations (9).

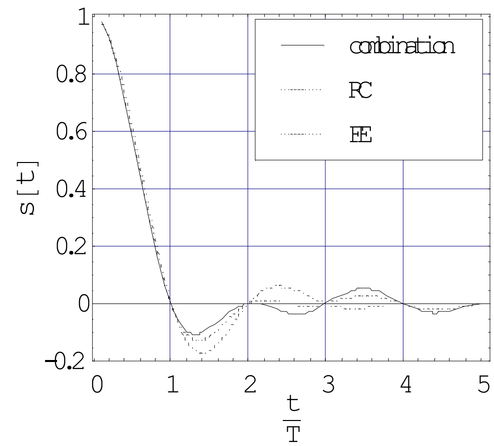



**Figure 3. Time characteristics of liniar combinations ($q_{opt}$=1.59), *raised-cosine* (RC) pulse and *flipped-exponential* (FE) pulse for an excess bandwidth α =0.35**

$$s_1(t) = \frac{\sin(2\pi t)\sin(2\pi\alpha t)}{4\alpha\pi^2 t^2}$$
$$s_2(t) = s_1(t)\left(2 - \frac{\tan(\pi\alpha t)}{\pi\alpha t}\right) \tag{9}$$
$$s_3(t) = s_1(t)\left(3 - \frac{3}{2\pi^2\alpha^2 t^2} + \frac{3\cot(2\pi\alpha t)}{\pi\alpha t}\right)$$
$$s_4(t) = s_1(t)\frac{4x^3 - 6x + (6x^2-3)\cot(2x) + 3\csc(2x)}{x^3}$$

where $x = \pi\alpha t$ .

This new family of Nyquist pulses shows reduced maximum distortion, a more open receiver eye and decreased symbol error probability in the presence of timing error, as compared with the *flipped-exponential* (FE) pulse [2] with the same roll-off factor. Its transmission properties were thoroughly investigated and show that the pulses have practical importance.

## 2. NEW LINEAR COMBINATION OF PULSES

In this paper we proposed the linear combination between two pulses $p_1(t)$ and $p_2(t)$ [6].

In the first case, we apply the linear combination technique to the pair of pulses built from the *flipped-exponential* (FE) pulse for $p_1(t)$ and $s_{i=2,3,4}(t)$ for $p_2(t)$.

Figure 4 illustrates the Nyquist filter characteristics [5] for $i = 2, 3, and\ 4$ together with the *flipped exponential* (FE) pulse defined in [2], taken as a reference.

The new pulses obtained as a result of the linear combination is

$$p_i(t) = qp_{FE}(t) + (1-q)s_i(t), \qquad i = 2,3,4 \tag{10}$$

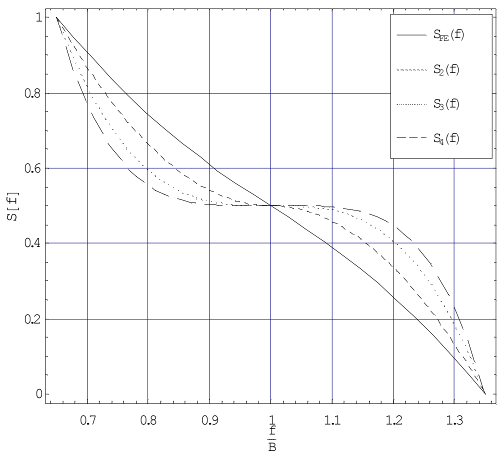



**Figure 4. Frequency characteristics for an excess bandwith α = 0.35**

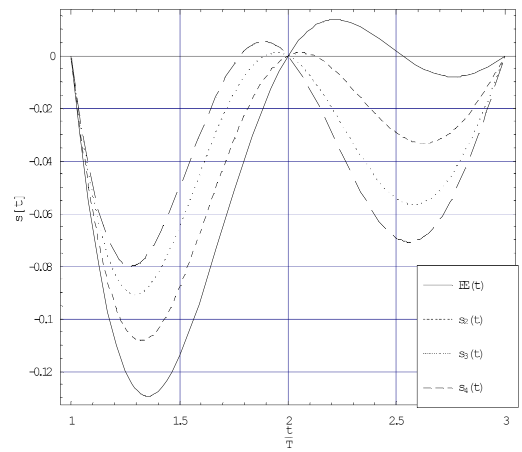



**Figure 5. Impulse response for the constituent pulses for α = 0.35**

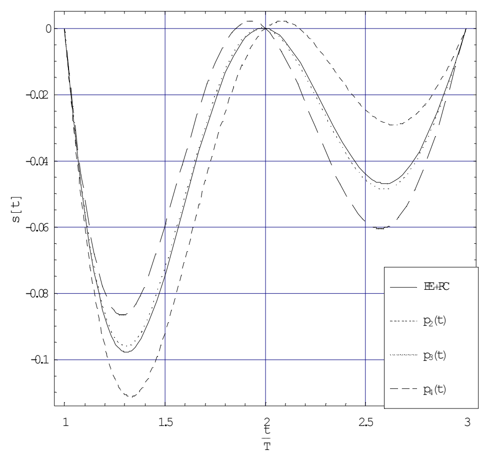



**Figure 6. The resulting pulses of linear combinations proposed (q=0.15) and FE+RC for α = 0.35**

The result of the combination is expected to have a more concave characteristic than the FE pulse for $q < 1$ and an improvement over the results presented in [6] and [5].

Figure 5 presents impulse responses for the *flipped-exponential* (FE) pulse and $s_i(t)$, $i = 2,3,4$.

A look at the figure 6 illustrates that the new pulses defined with (9) for i=3 follows closely the pulse proposed in [6]. Regarding the next pulse (i =4), though the decrease of the first side lobe is more significant, the side lobes are significantly larger, which results in increased ISI. The behaviour is similar to that of FE pulse [2] where increasing the roll-off factor $\alpha$ results in a decreased error probability.

In the second case, the pair of ISI-free pulses is built from $s_{i=2,3,4}(t)$ for $p_1(t)$ and *raised-cosine* (RC) pulse for $p_2(t)$.

Figure 7 illustrates the Nyquist filter characteristics [5] for $i = 2,3, and\ 4$ together with the *raised-cosine* (RC) pulse defined in [1], [2], taken as a reference.

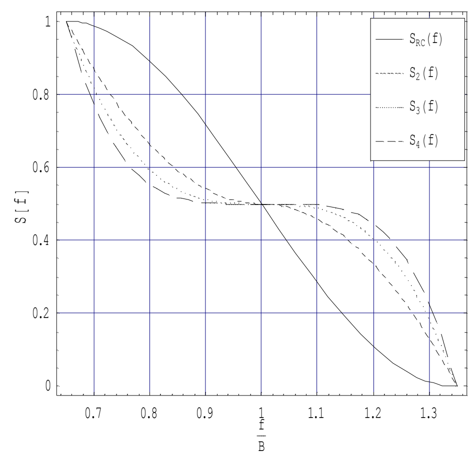



**Figure 7. Frequency characteristics for an excess bandwith α = 0.35**

The new pulses obtained as a result of the linear combination is

$$r_i(t) = qs_i(t) + (1-q)p_{RC}(t), \qquad i = 2,3,4 \qquad (11)$$

Figure 8 presents impulse responses for *raised-cosine* (RC) pulse and $s_{i=2,3,4}(t)$.

We observe, from the figure 9 that new pulse defined by (10) for *i* = 3 follows closely the pulse proposed in [6] and for *i* = 4 the decrease of the first side lobe is more significant and the side lobes are significantly larger.

The behaviour for the both cases discussed is similar to that of FE pulse [2] where increasing α results in decreased error probability. Their behaviour around $t/T = 3,4,\cdots$ is more flat, which accounts for their better properties regarding the error probability when sampled with a small time offset.

In the third case we observed and studied the linear combination between a new pulse shape c(t) and FE and c(t) and RC.

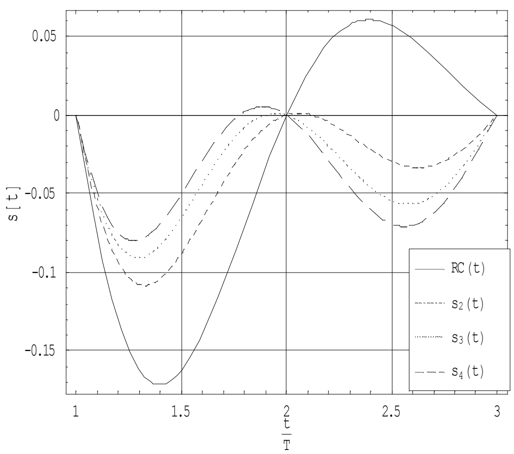



**Figure 8. Impulse response for the constituent pulses for α = 0.35**

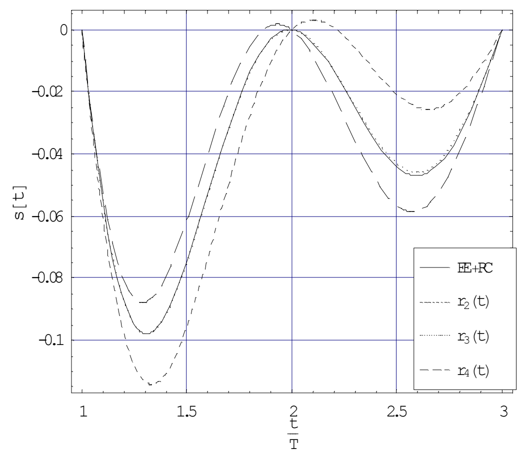



**Figure 9. The resulting pulses of linear combinations proposed (q=0.9) and FE+RC for α = 0.35**

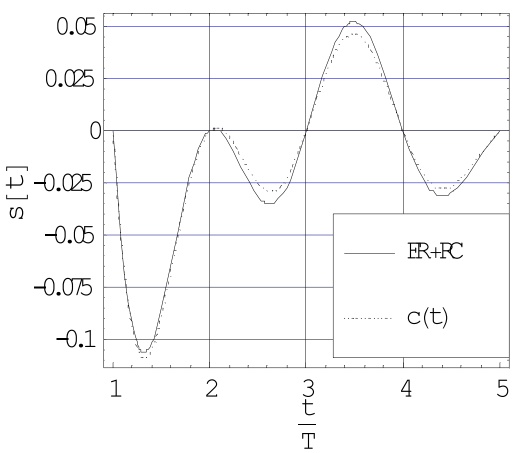



**Figure 10. The resulting pulses of linear combination FE+RC for α = 0.35, q=1.59 and c(t) for α =0.35**

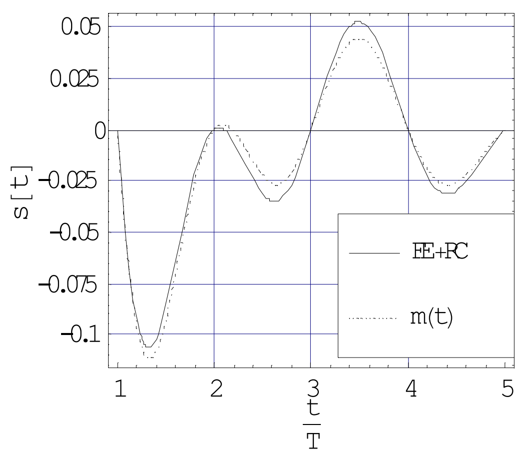



**Figure 11. The resulting pulses of linear combinations proposed m(t) for α = 0.35, q=0.11 and FE+RC for α = 0.35, q=1.59**

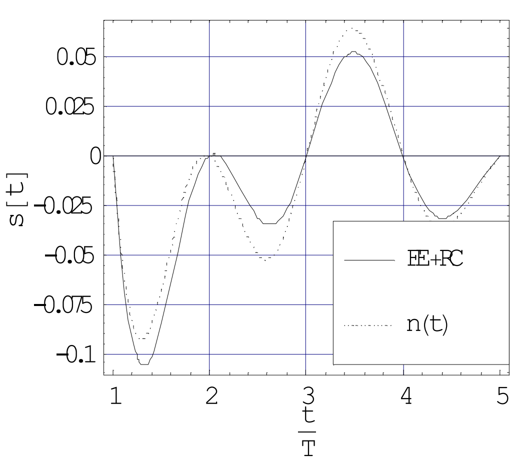



**Figure 12. The resulting pulses of linear combinations proposed n(t) for α = 0.35, q=1.3 and FE+RC for α = 0.35, q=1.59**

$$c(t)=\frac{\left(\dfrac{4}{t^2}+\dfrac{8a^2\pi\left(-3(3c+d)+a^2(c+3d)t^2\right)}{9-10a^2t^2+a^4t^4}\right)\sin(\pi t)\sin(a\pi t)}{4a\pi^2} \tag{12}$$

$$m(t)=qp_{FE}(t)+(1-q)c(t) \tag{13}$$

$$n(t)=qc(t)+(1-q)p_{RC}(t) \tag{14}$$

Figure 10 presents impulse response for linear combination between *raised-cosine* (RC) pulse with *flipped-exponential* (FE) pulse [6] and *c(t)* pulse.
In figure 11 and figure 12 are illustrated the linear combinations between *flipped-exponential* (FE) pulse with *c(t)* pulse, respective *raised-cosine* (RC) pulse with *c(t)* compared with combination reported in [6].

## 3. NUMERICAL RESULTS FOR ERROR PROBABILITY

Figure 13 and figure 14 show receiver eye diagrams for the linear combinations proposed compared with the linear combination reported in [6]. When receiver eye is sampled off center, as in practical receivers, timing error results in an increase of the average symbol error probability [2], [3], [13].

The error probability is calculated using the method of [13] for all proposed pulses and illustrated in Table 1, Table 2 and Table 3 together with those for the pulse proposed in [6] for all pulse shapes proposed and discussed previous.

We studied and explored further, the sensitivity of performance of combinations pulses to mismatch in SNR. The calculation of $q_{opt}$ needs the knowledge of SNR. In table 4, table 5 and table 6 are listed the results for different values of SNR. Usually, the sensitivity of combination pulses to SNR mismatch is a function of $\alpha$ .

## 4. CONCLUSIONS

A new family of Nyquist pulses obtained from a linear combination of two ISI-free pulses produced by Nyquist filters characteristics was proposed and investigated.

The result of the combination has a more concave characteristic, which transfers some energy into the high-frequency range and decreases the first side lobe in the time domain. This together with a decrease of the slope around the sampling time instants results in decreased error probability.

The last assertion is valid only for moderate timing errors and smaller roll-off factors ($\alpha$ <0.3), as the amplitude of the side lobes can be larger for $t/T > 0.2$ .

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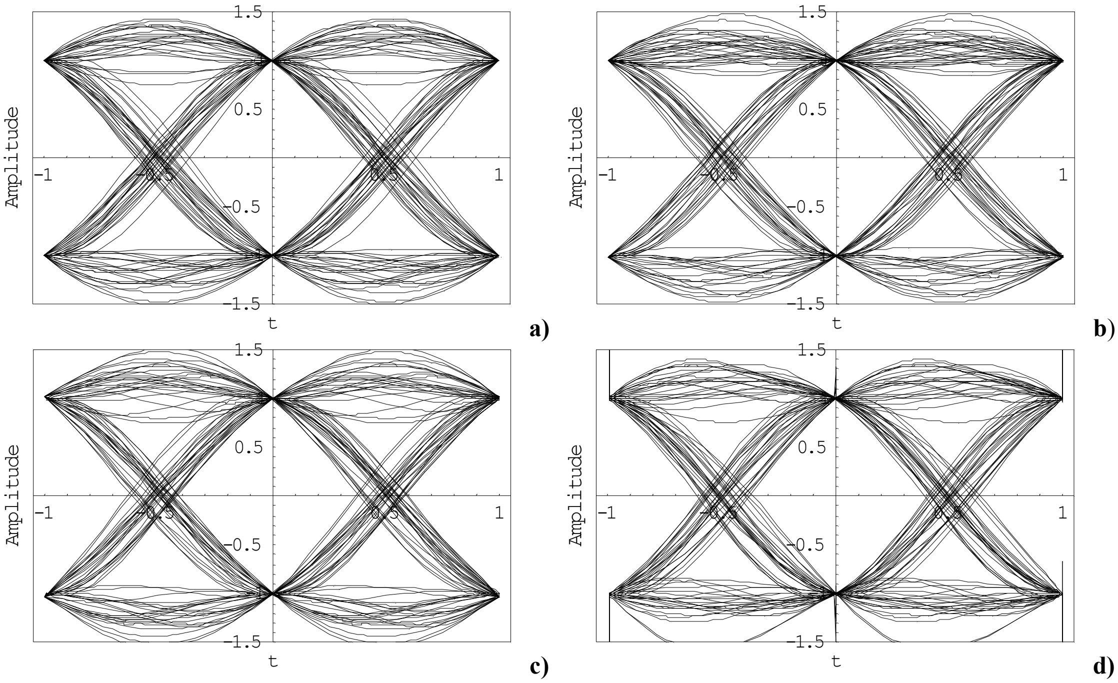


**Figure 10. Eye diagram for liniar combination:**
**FE+RC, α = 0.35, $q_{opt}$=1.59 (a); FE+$s_2(t)$, α = 0.35, $q_{opt}$=1.15 (b);**
**FE+$s_3(t)$, α = 0.35, $q_{opt}$=1 .15(c); FE+$s_4(t)$, α = 0.35, $q_{opt}$=1.15 (d);**

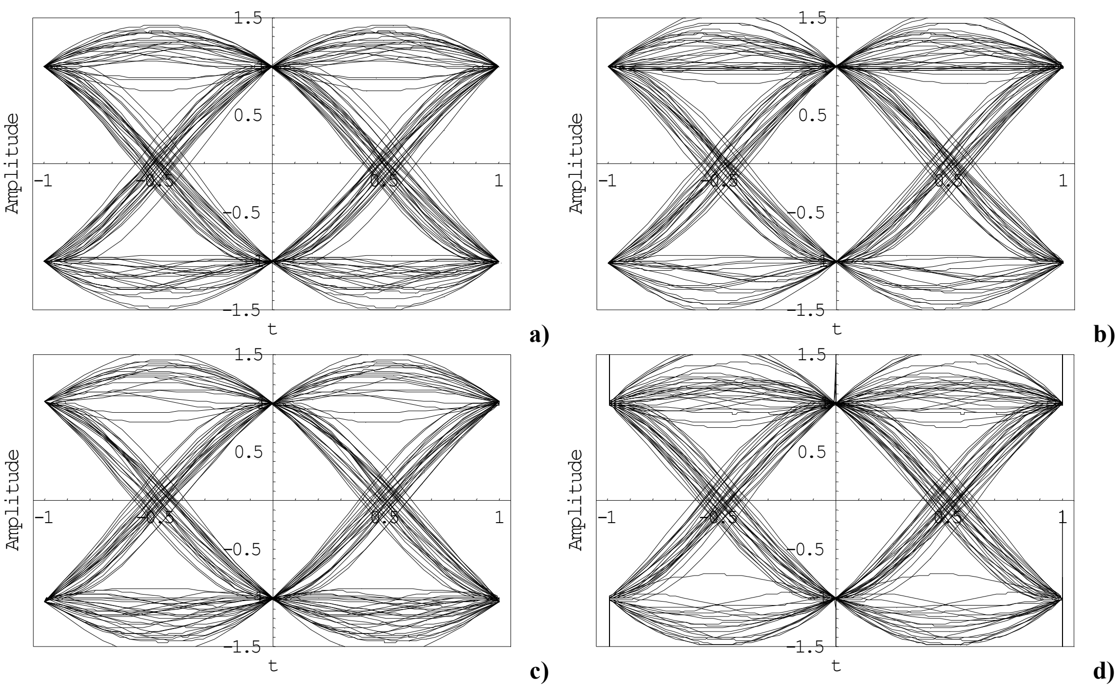


**Figure 11. Eye diagram for liniar combination:**
**FE+RC, α = 0.35, $q_{opt}$=1.59 (a); $s_2(t)$+RC, α = 0.35, q = 0.9 (b);**
**$s_3(t)$+RC, α = 0.35, q=0.9(c); $s_4(t)$+RC, α = 0.35, q=0.9 (d);**

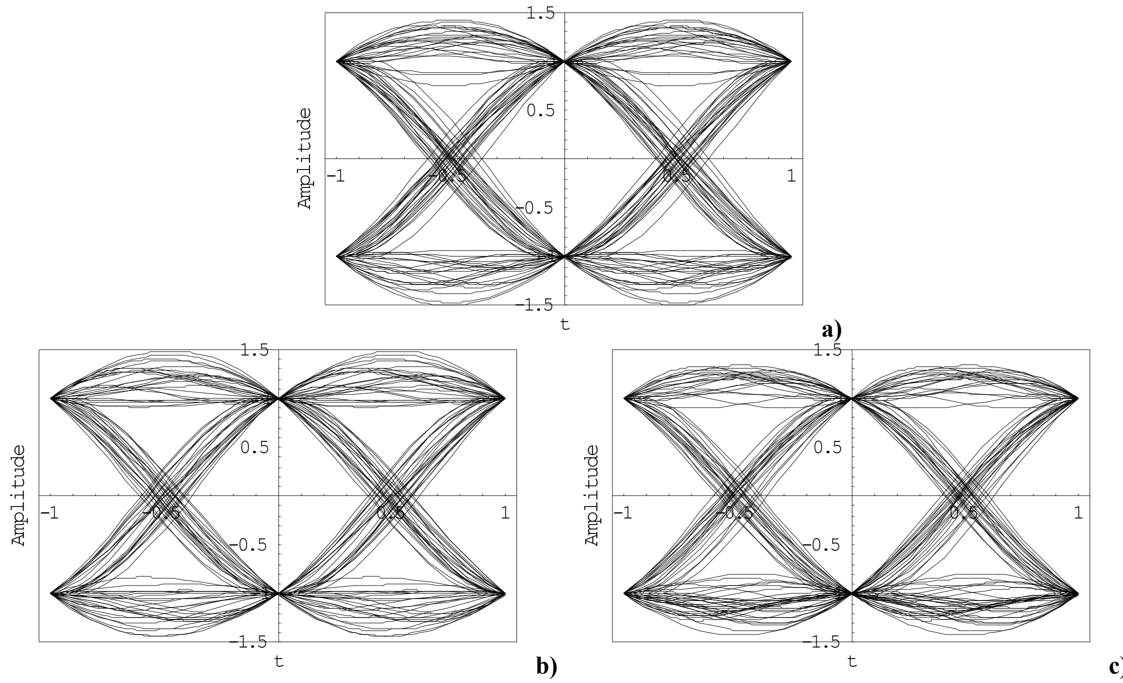


**Figure 12. Eye diagram for liniar combination:**
**FE+RC, α = 0.35, $q_{opt}$=1.59 (a);**
**FE+c(t), α = 0.35, q = 0.11 (b); c(t)+RC, α = 0.35, q=1.3(c);**

**Table 1: ISI error probability of new pulses discussed in the first case for N=$2^{10}$ interfering symbols and SNR = 15dB**

| α | $q_{opt}$ | $p_{FE}(t)+p_{RC}(t)$ | $q$ | $t/T$ | $p_{FE}(t)+s_2(t)$ | $p_{FE}(t)+s_3(t)$ | $p_{FE}(t)+s_4(t)$ |
|---|---|---|---|---|---|---|---|
| 0.25 | 1.82 | **$5.11871*10^{-8}$** | 0.25 | 0.05 | $5.38141*10^{-8}$ | $5.108*10^{-8}$ | $4.97109*10^{-8}$ |
| | | **$1.02678*10^{-6}$** | | 0.1 | $1.11282*10^{-6}$ | $1.01342*10^{-6}$ | $9.7076*10^{-7}$ |
| | | **$2.69726*10^{-4}$** | | 0.2 | $2.93066*10^{-4}$ | $2.6288*10^{-4}$ | $2.51771*10^{-4}$ |
| | | **$2.26367*10^{-3}$** | | 0.25 | $2.46029*10^{-3}$ | $2.21825*10^{-3}$ | $2.12458*10^{-3}$ |
| | | | 0.15 | 0.05 | $5.34224*10^{-8}$ | $5.0681*10^{-8}$ | $4.943*10^{-8}$ |
| | | | | 0.1 | $1.09893*10^{-6}$ | $1.00556*10^{-6}$ | $9.72702*10^{-7}$ |
| | | | | 0.2 | $2.88987*10^{-4}$ | $2.62394*10^{-4}$ | $2.55818*10^{-4}$ |
| | | | | 0.25 | $2.242676*10^{-3}$ | $2.20843*10^{-3}$ | $2.14681*10^{-3}$ |
| 0.35 | 1.59 | **$3.50275*10^{-8}$** | 0.35 | 0.05 | $3.63906*10^{-8}$ | $3.45609*10^{-8}$ | $3.36332*10^{-8}$ |
| | | **$4.4563*10^{-7}$** | | 0.1 | $4.7132*10^{-7}$ | $4.35722*10^{-7}$ | $4.2097*10^{-7}$ |
| | | **$8.28323*10^{-5}$** | | 0.2 | $8.64811*10^{-5}$ | $8.09469*10^{-5}$ | $7.97477*10^{-5}$ |
| | | **$7.69377*10^{-4}$** | | 0.25 | $8.02757*10^{-4}$ | $7.52566*10^{-4}$ | $7.41348*10^{-4}$ |
| | | | 0.15 | 0.05 | $3.58481*10^{-8}$ | $3.41116*10^{-8}$ | $3.34503*10^{-8}$ |
| | | | | 0.1 | $4.62577*10^{-7}$ | $4.38285*10^{-7}$ | $4.37232*10^{-7}$ |
| | | | | 0.2 | $8.58395*10^{-5}$ | $8.55366*10^{-5}$ | $8.99052*10^{-5}$ |
| | | | | 0.25 | $7.96348*10^{-4}$ | $7.91724*10^{-4}$ | $8.28778*10^{-8}$ |
| 0.5 | 1.41 | **$2.20747*10^{-8}$** | 0.5 | 0.05 | $2.26843*10^{-8}$ | $2.16942*10^{-8}$ | $2.11602*10^{-8}$ |
| | | **$1.61239*10^{-7}$** | | 0.1 | $1.68229*10^{-7}$ | $1.56997*10^{-7}$ | $1.50956*10^{-7}$ |
| | | **$1.9624*10^{-5}$** | | 0.2 | $1.98282*10^{-5}$ | $1.96072*10^{-5}$ | $1.96667*10^{-5}$ |
| | | **$1.95695*10^{-4}$** | | 0.25 | $1.94875*10^{-4}$ | $1.98046*10^{-4}$ | $2.03444*10^{-4}$ |
| | | | 0.15 | 0.05 | $2.2166*10^{-8}$ | $2.14598*10^{-8}$ | $2.13992*10^{-8}$ |
| | | | | 0.1 | $1.65417*10^{-7}$ | $1.632*10^{-7}$ | $1.67299*10^{-7}$ |
| | | | | 0.2 | $2.11378*10^{-5}$ | $2.49634*10^{-5}$ | $2.94045*10^{-5}$ |
| | | | | 0.25 | $2.12283*10^{-4}$ | $2.67958*10^{-4}$ | $3.32087*10^{-4}$ |

**Table 2: ISI error probability of the new pulses discussed in the second case for $N=2^{10}$ interfering symbols and SNR = 15 dB**

| α | $t/T$ | $q_{opt}$ | $p_{FE}(t)+p_{RC}(t)$ | $q_{opt}$ | $s_2(t)+p_{RC}(t)$ | $q_{opt}$ | $s_3(t)+p_{RC}(t)$ | $q_{opt}$ | $s_4(t)+p_{RC}(t)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.25 | 0.05 | 1.82 | **$5.11871*10^{-8}$** | 1.3 | $5.13466*10^{-8}$ | 1.1 | $5.02244*10^{-8}$ | 0.9 | $4.93926*10^{-8}$ |
| | 0.1 | | **$1.02678*10^{-6}$** | 1.2 | $1.05246*10^{-6}$ | 0.9 | $1.00651*10^{-6}$ | 0.9 | $9.66525*10^{-7}$ |
| | 0.2 | | **$2.69726*10^{-4}$** | 1.1 | $2.79607*10^{-4}$ | 0.9 | $2.61501*10^{-4}$ | 0.8 | $2.47915*10^{-4}$ |
| | 0.25 | | **$2.26367*10^{-3}$** | 1.2 | $2.3408*10^{-3}$ | 0.9 | $2.20498*10^{-3}$ | 0.8 | $2.10399*10^{-3}$ |
| 0.35 | 0.05 | 1.59 | **$3.50275*10^{-8}$** | 1.2 | $3.49256*10^{-8}$ | 0.9 | $3.41446*10^{-8}$ | 0.9 | $3.33515*10^{-8}$ |
| | 0.1 | | **$4.4563*10^{-7}$** | 1.1 | $4.58391*10^{-7}$ | 0.8 | $4.34889*10^{-7}$ | 0.8 | $4.15586*10^{-7}$ |
| | 0.2 | | **$8.28323*10^{-5}$** | 0.9 | $8.67883*10^{-5}$ | 0.8 | $7.97962*10^{-5}$ | 0.7 | $7.46804*10^{-5}$ |
| | 0.25 | | **$7.69377*10^{-4}$** | 0.9 | $8.05138*10^{-4}$ | 0.8 | $7.42701*10^{-4}$ | 0.7 | $6.97587*10^{-4}$ |
| 0.5 | 0.05 | 1.41 | **$2.20747*10^{-8}$** | 1.1 | $2.1986*10^{-8}$ | 0.9 | $2.13932*10^{-8}$ | 0.8 | $2.09081*10^{-8}$ |
| | 0.1 | | **$1.61239*10^{-7}$** | 0.9 | $1.66776*10^{-7}$ | 0.8 | $1.55559*10^{-7}$ | 0.7 | $1.47297*10^{-7}$ |
| | 0.2 | | **$1.9624*10^{-5}$** | 0.8 | $2.0222*10^{-5}$ | 0.6 | $1.86363*10^{-5}$ | 0.6 | $1.7228*10^{-5}$ |
| | 0.25 | | **$1.95695*10^{-4}$** | 0.8 | $1.98145*10^{-4}$ | 0.6 | $1.82029*10^{-4}$ | 0.6 | $1.712*10^{-4}$ |

**Table 3: ISI error probability of the new pulses discussed in the third case for $N=2^{10}$ interfering symbols and SNR = 15 dB**

| α | t/T | $q_{opt}$ | $p_{FE}(t)+p_{RC}(t)$ | q | $p_{FE}(t)+c(t)$ | q | $c(t)+p_{RC}(t)$ |
|---|---|---|---|---|---|---|---|
| 0.25 | 0.05 | 1.81 | **$5.11871*10^{-8}$** | 0.11 | $5.30749*10^{-8}$ | 1.2 | $5.03982*10^{-8}$ |
| | 0.1 | | **$1.02678*10^{-6}$** | 0.11 | $1.0792*10^{-6}$ | 1.1 | $1.01345*10^{-6}$ |
| | 0.2 | | **$2.69726*10^{-4}$** | 0.11 | $2.80928*10^{-4}$ | 1.1 | $2.65528*10^{-4}$ |
| | 0.25 | | **$2.26367*10^{-3}$** | 0.11 | $2.36986*10^{-3}$ | 1.1 | $2.23363*10^{-3}$ |
| 0.35 | 0.05 | 1.59 | **$3.50275*10^{-8}$** | 0.11 | $3.552*10^{-8}$ | 1.4 | $3.43214*10^{-8}$ |
| | 0.1 | | **$4.4563*10^{-7}$** | 0.11 | $4.50198*10^{-7}$ | 1.3 | $4.3966*10^{-7}$ |
| | 0.2 | | **$8.28323*10^{-5}$** | 0.11 | $8.17282*10^{-5}$ | 1.2 | $8.23268*10^{-5}$ |
| | 0.25 | | **$7.69377*10^{-4}$** | 0.11 | $7.60435*10^{-4}$ | 1.2 | $7.64369*10^{-4}$ |
| 0.5 | 0.05 | 1.41 | **$2.20747*10^{-8}$** | 0.11 | $2.18832*10^{-8}$ | 1.2 | $2.15573*10^{-8}$ |
| | 0.1 | | **$1.61239*10^{-7}$** | 0.11 | $1.58896*10^{-7}$ | 1.1 | $1.59645*10^{-7}$ |
| | 0.2 | | **$1.9624*10^{-5}$** | 0.41 | $1.88507*10^{-5}$ | 0.9 | $1.92399*10^{-5}$ |
| | 0.25 | | **$1.95695*10^{-4}$** | 0.51 | $1.84949*10^{-4}$ | 0.8 | $1.8595*10^{-4}$ |

**Table 4: ISI error probability of the new pulses discussed in the first case for fixed timing error *t/T=0.15***

| *SNR* | $\alpha$ | $q_{opt}$ | $p_{FE}(t)+p_{RC}(t)$ | $q$ | $p_{FE}(t)+s_2(t)$ | $p_{FE}(t)+s_3(t)$ | $p_{FE}(t)+s_4(t)$ |
|---|---|---|---|---|---|---|---|
| 5dB | 0.25 | 2.20 | **$5.28969*10^{-2}$** | 0.25 | $5.32554*10^{-2}$ | $5.2961*10^{-2}$ | $5.28075*10^{-2}$ |
| | | | | 0.15 | $5.32132*10^{-2}$ | $5.29133*10^{-2}$ | $5.27699*10^{-2}$ |
| | 0.35 | 1.97 | **$5.10245*10^{-2}$** | 0.35 | $5.12616*10^{-2}$ | $5.10206*10^{-2}$ | $5.09035*10^{-2}$ |
| | | | | 0.15 | $5.11892*10^{-2}$ | $5.09694*10^{-2}$ | $5.09027*10^{-2}$ |
| | 0.5 | 1.65 | **$4.93634*10^{-2}$** | 0.5 | $4.9362*10^{-2}$ | $4.92877*10^{-2}$ | $4.92721*10^{-2}$ |
| | | | | 0.15 | $4.93521*10^{-2}$ | $4.9468*10^{-2}$ | $4.96612*10^{-2}$ |
| 10dB | 0.25 | 2.20 | **$4.22034*10^{-3}$** | 0.25 | $4.40167*10^{-3}$ | $4.25057*10^{-3}$ | $4.17259*10^{-3}$ |
| | | | | 0.15 | $4.38001*10^{-3}$ | $4.22668*10^{-3}$ | $4.15397*10^{-3}$ |
| | 0.35 | 1.97 | **$3.28069*10^{-3}$** | 0.35 | $3.40698*10^{-3}$ | $3.28957*10^{-3}$ | $3.22989*10^{-3}$ |
| | | | | 0.15 | $3.37178*10^{-3}$ | $3.2598*10^{-3}$ | $3.21801*10^{-3}$ |
| | 0.5 | 1.65 | **$2.43692*10^{-3}$** | 0.5 | $2.49474*10^{-3}$ | $2.43058*10^{-3}$ | $2.39827*10^{-3}$ |
| | | | | 0.15 | $2.46416*10^{-3}$ | $2.43664*10^{-3}$ | $2.45453*10^{-3}$ |
| 15dB | 0.25 | 2.20 | **$2.12668*10^{-5}$** | 0.25 | $2.1669*10^{-5}$ | $1.9462*10^{-5}$ | $1.86185*10^{-5}$ |
| | | | | 0.15 | $2.13694*10^{-5}$ | $1.93966*10^{-5}$ | $1.886*10^{-5}$ |
| | 0.35 | 1.97 | **$7.29942*10^{-6}$** | 0.35 | $6.9123*10^{-6}$ | $6.42029*10^{-6}$ | $6.27357*10^{-6}$ |
| | | | | 0.15 | $6.82797*10^{-6}$ | $6.67105*10^{-6}$ | $6.88947*10^{-6}$ |
| | 0.5 | 1.65 | **$1.8524*10^{-6}$** | 0.5 | $1.79069*10^{-6}$ | $1.71164*10^{-6}$ | $1.67304*10^{-6}$ |
| | | | | 0.15 | $1.83594*10^{-6}$ | $1.97652*10^{-6}$ | $2.16482*10^{-6}$ |

**Table 5: ISI error probability of the new pulses discussed in the second case for fixed timing error *t/T=15***

| *SNR* | $\alpha$ | $q_{opt}$ | $p_{FE}(t)+p_{RC}(t)$ | $q_{opt}$ | $s_2(t)+p_{RC}(t)$ | $q_{opt}$ | $s_3(t)+p_{RC}(t)$ | $q_{opt}$ | $s_4(t)+p_{RC}(t)$ |
|---|---|---|---|---|---|---|---|---|---|
| 5dB | 0.25 | 2.20 | **$5.28969*10^{-2}$** | 1.2 | $5.30019*10^{-2}$ | 0.9 | $5.2931*10^{-2}$ | 0.9 | $5.27685*10^{-2}$ |
| | 0.35 | 1.79 | **$5.10245*10^{-2}$** | 1.1 | $5.10936*10^{-2}$ | 0.8 | $5.10332*10^{-2}$ | 0.8 | $5.08825*10^{-2}$ |
| | 0.5 | 1.65 | **$4.93634*10^{-2}$** | 0.9 | $4.93544*10^{-2}$ | 0.8 | $4.92997*10^{-2}$ | 0.7 | $4.9228*10^{-2}$ |
| 10dB | 0.25 | 2.20 | **$4.22034*10^{-3}$** | 1.2 | $4.27308*10^{-3}$ | 0.9 | $4.23546*10^{-3}$ | 0.9 | $4.15304*10^{-3}$ |
| | 0.35 | 1.79 | **$3.28069*10^{-3}$** | 1.1 | $3.32232*10^{-3}$ | 0.8 | $3.29645*10^{-3}$ | 0.8 | $3.22032*10^{-3}$ |
| | 0.5 | 1.65 | **$2.43692*10^{-3}$** | 0.9 | $2.47595*10^{-3}$ | 0.8 | $2.41783*10^{-3}$ | 0.7 | $2.38664*10^{-3}$ |
| 15dB | 0.25 | 2.20 | **$2.12668*10^{-5}$** | 1.2 | $2.06887*10^{-5}$ | 0.9 | $1.93495*10^{-5}$ | 0.9 | $1.86415*10^{-5}$ |
| | 0.35 | 1.79 | **$7.29942*10^{-6}$** | 1.1 | $6.97076*10^{-6}$ | 0.8 | $6.35638*10^{-6}$ | 0.8 | $6.1519*10^{-6}$ |
| | 0.5 | 1.65 | **$1.8524*10^{-6}$** | 0.9 | $1.82442*10^{-6}$ | 0.8 | $1.73288*10^{-6}$ | 0.7 | $1.59612*10^{-6}$ |

**Table 6: ISI error probability of the new pulses discussed in the third case for fixed timing error *t/T=15***

| *SNR* | $\alpha$ | $q_{opt}$ | $p_{FE}(t)+p_{RC}(t)$ | $q_{opt}$ | $p_{FE}(t)+c(t)$ | $q_{opt}$ | $c(t)+p_{RC}(t)$ |
|---|---|---|---|---|---|---|---|
| 5dB | 0.25 | 2.20 | **$5.28969*10^{-2}$** | 0.11 | $5.31817*10^{-2}$ | 1.3 | $5.28953*10^{-2}$ |
| | 0.35 | 1.79 | **$5.10245*10^{-2}$** | 0.11 | $5.11478*10^{-2}$ | 1.1 | $5.10358*10^{-2}$ |
| | 0.5 | 1.65 | **$4.93634*10^{-2}$** | 0.11 | $4.92986*10^{-2}$ | 1.1 | $4.93522*10^{-2}$ |
| 10dB | 0.25 | 2.20 | **$4.22034*10^{-3}$** | 0.11 | $43629*10^{-3}$ | 1.3 | $4.21806*10^{-3}$ |
| | 0.35 | 1.79 | **$3.28069*10^{-3}$** | 0.11 | $3.35206*10^{-3}$ | 1.1 | $3.29611*10^{-3}$ |
| | 0.5 | 1.65 | **$2.43692*10^{-3}$** | 0.11 | $2.44241*10^{-3}$ | 1.1 | $2.42936*10^{-3}$ |
| 15dB | 0.25 | 2.20 | **$2.12668*10^{-5}$** | 0.11 | $2.08091*10^{-5}$ | 1.3 | $1.97701*10^{-5}$ |
| | 0.35 | 1.79 | **$7.29942*10^{-6}$** | 0.11 | $6.54464*10^{-6}$ | 1.1 | $6.51413*10^{-6}$ |
| | 0.5 | 1.65 | **$1.8524*10^{-6}$** | 0.11 | $1.72092*10^{-6}$ | 1.1 | $1.82937*10^{-6}$ |